MTH 201: Multivariable Calculus and Differential Equations

Homework III

(Due 16/09)

1. The introduction of polar coordinates changes $f(x, y)$ into $\phi(r, \theta)$, where $x = r\cos\theta$ and $y = r\sin\theta$. Express the second-order partial derivatives $\frac{\partial^2\phi}{\partial r^2}$, $\frac{\partial^2\phi}{\partial r\partial\theta}$, $\frac{\partial^2\phi}{\partial\theta\partial r}$, and $\frac{\partial^2\phi}{\partial\theta^2}$, in terms of partial derivatives of f.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $g : \mathbb{R}^3 \to \mathbb{R}^3$ be two vector fields defined as follows:

$$f(x, y, z) = (x^2 + y + z)i + (2x + y + z^2)j,$$

$$g(u, v, w) = uv^2w^2i + w^2\sin vj + u^2e^vk.$$

   (a) Compute each of the Jacobian matrices $Df(x, y, z)$ and $Dg(u, v, w)$.
   (b) Compute the composition $h(u, v, w) = f(g(u, v, w))$.
   (c) Compute the Jacobian matrix $Dh(u, 0, w)$.

3. Find all differential vector fields $f : \mathbb{R}^3 \to \mathbb{R}^3$ that satisfy the following conditions.

   (a) The Jacobian matrix $Df(x, y, z)$ is the identity matrix of order 3.
   (b) The Jacobian matrix $Df(x, y, z)$ is a diagonal matrix of the form $\text{diag}(p(x), q(y), r(z))$, where p, q, and r are given continuous functions.

4. The equations $u = f(x, y, z)$, $x = X(r, s, t)$, $y = Y(r, s, t)$, and $z = Z(r, s, t)$ define u as a function of r, s, and t, say $u = F(r, s, t)$.

   (a) Use chain rule to express $\frac{\partial F}{\partial r}$, $\frac{\partial F}{\partial s}$, and $\frac{\partial F}{\partial t}$ in terms of the partial derivatives of f, X, Y, and Z.
   (b) Solve (a) for: $X(r, s, t) = r + s + t$, $Y(r, s, t) = r - 2s + 3t$, $Z(r, s, t) = 2r + s - t$.
   (c) Solve (a) for: $X(r, s, t) = r^2 + s^2 + t^2$, $Y(r, s, t) = r^2 - s^2 - t^2$, $Z(r, s, t) = r^2 - s^2 + t^2$.

5. The change of variables $x = uv$, $y = \frac{1}{2}(u^2 - v^2)$ transforms $f(x, y)$ to $g(u, v)$.

   (a) Assuming the equality of mixed partials, compute $\frac{\partial g}{\partial u}$, $\frac{\partial g}{\partial v}$, $\frac{\partial^2 g}{\partial u\partial v}$ in terms of the partial derivatives of f.
   (b) If $\|\nabla f(x, y)\|^2 = 2$ for all x and y, determine constants a and b such that

$$a\left(\frac{\partial g}{\partial u}\right)^2 - b\left(\frac{\partial g}{\partial v}\right)^2 = u^2 + v^2.$$

6. Define $f(x, y) = \int_0^{\sqrt{xy}} e^{-t^2}\, dt$ for $x > 0$ and $y > 0$. Compute $\frac{\partial f}{\partial x}$ in terms of x and y.

7. Let the scalar field f be defined by

$$f(x, y) = \begin{cases} \dfrac{xy^3}{x^3 + y^6} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

   (a) Prove that $f'(0; a)$ exists for every vector a and compute its value in terms of the components of a.

(b) Determine whether or not f is continuous at origin.

8. Let $v = xi + yj + zk$ and $r = \|v\|$. If A and B are constant vectors, show that:

   (a) $A \cdot \nabla \left(\frac{1}{r}\right) = -\frac{A \cdot v}{r^3}$.

9. Find the equation of tangent plane to the following surfaces at (x_0, y_0, z_0).

   (a) The paraboloid $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

   (b) The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

10. Find the set of points at which the tangent planes to the two spheres $(x-a)^2 + (y-b)^2 + (z-c)^2 = 1$ and $x^2 + y^2 + z^2 = 1$ intersect orthogonally.